Loss Models, Assignment 1

Academic Year 2019-2020

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Instructions for the Assignments

You should provide answers in English. Motivate all your answers extensively, you get points for the methods you use and for your clear explanation of them, not just for the final answer.

Success!

Deliverables for the Assignments

Please hand in on or before November, 5 via `TOLEDO`:

- A single document containing your solutions. Handwritten solutions are accepted, but then you should scan your solutions and combine these scans into a single document. Software to do this (for free) is available online.

Please mention the names and student numbers of your team members. It is allowed to work in teams (with two students maximum); it suffices to submit one solution per team. Grades and feedback will be provided through TOLEDO. We provide more effective feedback when you additionally hand in a printed or written solution during the class of October, 31 or November, 5.

Assignment Questions

Exercise 1.
The loss distribution for an insurance portfolio is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - 0.5 \cdot e^{-\beta \cdot x} & x \geq 0 \end{cases},$$

for some $\beta > 0$. Calculate for this random variable (as a function of β):

1. survival function, $S(x)$
2. the probability of having no loss, $P(X = 0)$
3. mean, μ

4. standard deviation, σ
5. expected loss with a deductible of 30, $E((X-30)_+)$
6. limited loss with a limit of 30, $E(X \wedge 30)$
7. excess of loss at 30, $e_X(30)$
8. 95% Value at Risk, $VaR_{0.95}(X)$

Solution 1.

1. $$S(x) = 1 - F(x) = \begin{cases} 1 & x < 0 \\ 0.5 \cdot e^{-\beta \cdot x} & x \geq 0 \end{cases}$$

2. This probability is the size of the jump in the distribution function at $x = 0$. In $x = 0$ the distribution function F jumps from 0 to 0.5, therefore $P(X = 0) = 0.5$.

3. The mean is computed as
$$\begin{aligned} E(X) &= 0 \cdot 0.5 + \int_0^\infty x \cdot 0.5 \cdot \beta \cdot e^{-\beta \cdot x}\, dx \\ &= 0 + \beta \cdot 0.5 \cdot \left(\frac{-x \cdot e^{-\beta \cdot x}}{\beta} \bigg|_0^\infty + \int_0^\infty \frac{e^{-\beta \cdot \cdot x}}{\beta} \right) && \text{(Partial Integration)} \\ &= \beta \cdot 0.5 \cdot \frac{e^{-\beta \cdot x}}{\beta^2} \bigg|_0^\infty = \frac{0.5}{\beta}. \end{aligned}$$

4. The variance can be found as
$$Var(X) = E(X^2) - E(X)^2.$$
We already computed $E(X)$, so we continue to find $E(X^2)$.
$$\begin{aligned} E(X^2) &= 0^2 \cdot 0.5 + \int_0^\infty x^2 \cdot 0.5 \cdot \beta \cdot e^{-\beta \cdot x}\, dx \\ &= 0 + \beta \cdot 0.5 \cdot \left(\frac{-x^2 \cdot e^{-\beta \cdot x}}{\beta^2} \bigg|_0^\infty + \int_0^\infty \frac{2 \cdot x \cdot e^{-\beta \cdot x}}{\beta} \right) && \text{(Partial Integration)} \\ &= \frac{2}{\beta} \int_0^\infty 0.5 \cdot \beta \cdot x \cdot e^{-\beta \cdot x}\, dx = \frac{2}{\beta} E(X) = \frac{1}{\beta^2}. \end{aligned}$$
The standard deviation is
$$\sigma = \sqrt{E(X^2) - E(X)^2} = \sqrt{\frac{1}{\beta^2} - \frac{0.5^2}{\beta^2}} = \frac{\sqrt{0.75}}{\beta}.$$

5. The expected loss with a deductible of 30 is,
$$\begin{aligned} E((X-30)_+) &= 0 \cdot 0.5 + \int_{30}^\infty (x - 30) \cdot 0.5 \cdot \beta \cdot e^{-\beta \cdot x}\, dx \\ &= 0 + \beta \cdot 0.5 \cdot \left(\frac{-(x-30) \cdot e^{-\beta \cdot x}}{\beta} \bigg|_{30}^\infty + \int_{30}^\infty \frac{e^{-\beta \cdot x}}{\beta} \right) && \text{(Partial Integration)} \\ &= \beta \cdot 0.5 \cdot \frac{e^{-\beta \cdot x}}{\beta^2} \bigg|_{30}^\infty = \frac{0.5 \cdot e^{-30 \cdot \beta}}{\beta}. \end{aligned}$$

6. For the limited expected loss, we use the identity

$$E((X-30)_+) + E(X \wedge 30) = E(X).$$

Therefore

$$E(X \wedge 30) = E(X) - E((X-30)_+) = \frac{0.5}{\beta} - \frac{0.5 \cdot e^{-30 \cdot \beta}}{\beta} = \frac{0.5}{\beta} \cdot (1 - e^{-30 \cdot \beta}).$$

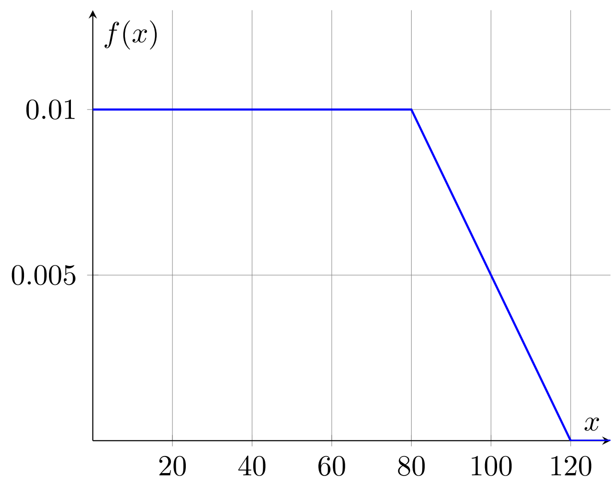
7. The excess of loss is

$$e_Y(30) = \frac{E(X-30)_+}{S(30)} = \frac{0.5 \cdot e^{-30 \cdot \beta}}{\beta \cdot 0.5 \cdot e^{-\beta \cdot 30}} = \frac{1}{\beta}.$$

8. Value at risk is defines such that $S(VaR_p(Y)) = 1 - p$. With $p = 0.95$, we solve

$$0.05 = S(VaR_{0.95}(Y)) = 0.5 \cdot e^{-\beta \cdot x} \Rightarrow x = -\frac{\log(0.1)}{\beta}$$

Exercise 2.
(Exam question, August 2017) The graph of the density function for losses is:



Calculate the Loss Elimination Ratio for an ordinary deductible of 20. The LER is the ratio of the decrease in the expected payment with an ordinary deductible to the expected payment without the deductible. Explain intuitively what this ratio expresses.

Solution 2.
The density function is

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 0.01 & 0 < x \leq 80 \\ 0.03 - 0.00025 \cdot x & 80 < x \leq 120 \\ 0 & x > 120 \end{cases}$$

Let X be the loss random variable, then the expected loss is

$$\begin{aligned} E(X) &= \int_0^{80} 0.01 \cdot x \, dx + \int_{80}^{120} 0.03 \cdot x - 0.00025 \cdot x^2 \, dx \\ &= 0.005x^2 \Big|_0^{80} + 0.015x^2 \Big|_{80}^{120} - \frac{0.00025}{3} x^3 \Big|_{80}^{120} \\ &= 32 + 120 - 101.33 = 50.67 \end{aligned}$$

and

$$\begin{aligned} E((X-20)_+) = E(X) - \int_0^{20} 0.01 \cdot x - 20 \cdot P(X > 20) \\ &= 50.67 - 0.005x^2 \Big|_0^{20} - 20 \cdot 0.8 \\ &= 50.67 - 2 - 16 = 32.67. \end{aligned}$$

The loss elimination ration is

$$\text{LER} = \frac{50.67 - 32.67}{50.67} = 35.53\%.$$

This is the reduction in the total loss for the insurer when introducing this deductible. This implies that 35.53% of the risk remains with the insured.

Exercise 3.
Join the Loss Models group on DataCamp (invitation sent to the email address you use in TOLEDO) and solve chapters 1, 2 and 4 of the course 'Foundations of probability in R'. Your progress will be monitored in the teacher's dashboard that comes with the group. When you have never used R before, we recommend you to also have a look at the course 'Introduction to R'.